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THERMAL FIELD ON THE SURFACE OF A MATERIAL HEATED BY A LASER PULSE IN A NARROW RING OF ILLUMINATION

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Based on a solution of the problem of the temperature effect of instantaneous heat sources uniformly distributed in a thin plane ring on the boundary of a semiinfinite body and different approximations of that solution, we constructed a dynamic model of a thermal field on the surface of a material heated by a laser pulse in a narrow ring of illumination.

Statement of the Problem. Initial Equations. A two-dimensional nonstationary heat conduction problem for a semi-infinite body with heat supply to it through a thin plane ring on the surface was solved in [1] (see also [2]). The solution is represented by double series whose terms are determined by the convolution of source function with a complex mathematical expression containing Whittaker functions of different orders. In the present work we seek possible approximations of this solution as applied to the conditions of material heating by short pulses of laser radiation focused in a narrow ring of illumination.

Let R_1 and R_2 be the inner and outer radii of the ring within which a plane heat source of specific power $q(\tau)$ acts from the time moment $\tau = 0$. A general solution in the image space for the region 1 $(0 \le r < R_1, z \ge 0)$ can be written in form

$$\overline{\Theta}_1(r, z, s) = 2\frac{\overline{q}(s)}{\pi\lambda}\int_{\infty}^{0}\frac{\cos(pz)}{\sqrt{p^2+s/a}}I_0\left(r\sqrt{p^2+s/a}\right)\times$$

$$\times\left[R_1K_1\left(R_1\sqrt{p^2+s/a}\right)-R_2K_1\left(R_2\sqrt{p^2+s/a}\right)\right]dp\,,$$

where $\overline{\Theta}(r, z, s)$ is the Laplace transform of $\Theta(r, z, \tau) = T(r, z, \tau) - T_0$ with respect to τ and $\overline{q}(s)$ is the Laplace transform of $q(\tau)$. The solutions for regions 2 $(R_1 \le r \le R_2, z \ge 0)$ and 3 $(R_2 < r < \infty, z \ge 0)$ are similar. For points of the body that lie on the surface $(z = 0)$, transition from the space of transforms to the space of inverse transforms is possible in analytic form. The solution takes the form [1]

$$\Theta_1(r, 0, \tau) = \sqrt{2/\pi}\,\frac{1}{b}\sum_{n=0}^{\infty}\sum_{m=0}^{n}A_{n,m}\left(\frac{r}{R_2}\right)^{2n}\left(\frac{R_2}{\sqrt{a}}\right)^{2n-m-1/2}\times$$

$$\times\int_0^{\tau}q(\tau-\xi)\,\xi^{-n+m/2-1/4}\left[\left(\frac{R_1}{R_2}\right)^{-m-1/2}\exp\left(-\frac{R_2^2}{8a\xi}\right)\times\right.$$

$$\times W_{\frac{2n-m}{2}+1/4,\,m/2-1/4}\left(\frac{R_1^2}{4a\xi}\right)-\exp\left(-\frac{R_2^2}{8a\xi}\right)\times$$

$$\left.\times W_{\frac{2n-m}{2},\,m/2-1/4}\left(\frac{R_2^2}{4a\xi}\right)\right]d\xi\quad(0 \le r \le R_1)\,,$$

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$$\Theta_2(r,0,\tau) - \frac{1}{b\sqrt{\pi}}\int_0^{\tau}\frac{q(\tau-\xi)}{\sqrt{\xi}}d\xi - \sqrt{2/\pi}\,\frac{1}{b}\sum_{n=0}^{\infty}\sum_{m=0}^{n}A_{n,m}\times$$

$$\times\left(\frac{r}{R_2}\right)^{2n}\left(\frac{R_2}{\sqrt{a}}\right)^{2n-m-1/2}\int_0^{\tau}q(\tau-\xi)\,\xi^{-n+m/2-1/4}\times$$

$$\times\exp\left(-\frac{R_2^2}{8a\xi}\right)W_{\frac{2n-m}{2}+1/4,m/2-1/4}\left(\frac{R_2^2}{4a\xi}\right)d\xi - \sqrt{2/\pi}\,\frac{R_1}{\lambda}\times$$

$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{A_{n,m}}{2(n+1)}\left(\frac{R_1}{r}\right)\left(\frac{R_1}{\sqrt{a}}\right)^{2n}\left(\frac{r}{\sqrt{a}}\right)^{-m-1/2}\int_0^{\tau}q(\tau-\xi)\,\xi^{-n+m/2-3/4}\times$$

$$\times\exp\left(-\frac{r^2}{8a\xi}\right)W_{\frac{2n-m}{2}+3/4,m/2+1/4}\left(\frac{r^2}{4a\xi}\right)d\xi\quad(R_1\le r\le R_2), \tag{1}$$

$$\Theta_3(r,0,\tau) = \sqrt{2/\pi}\,\frac{R_2}{\lambda}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{A_{n,m}}{2(n+1)}\left(\frac{R_2}{r}\right)\left(\frac{R_2}{\sqrt{a}}\right)^{2n}\left(\frac{r}{\sqrt{a}}\right)^{-m-1/2}\times$$

$$\times\left[1-\left(\frac{R_1}{R_2}\right)^{2n+2}\right]\int_0^{\tau}q(\tau-\xi)\,\xi^{-n+m/2-3/4}\exp\left(-\frac{r^2}{8a\xi}\right)\times$$

$$\times W_{\frac{2n-m}{2}+3/4,m/2+1/4}\left(\frac{r^2}{4a\xi}\right)d\xi\quad(R_2<r<\infty).$$

Here, $W_{\nu,\mu}(\cdot)$ are Whittaker functions, $(2m-1)!! = 1\cdot3\cdot5\ldots\cdot(2m-1)$,

$$A_{n,m} = \frac{(2m-1)!!}{4^n n!\,m!\,(n-m)!}. \tag{2}$$

In [2] the integrals entering into (1) were calculated for two particular cases: $q(\tau) = U(\tau)$ and $q(\tau) = U(\tau) - U(\tau-\tau_0)$, where $U(\tau)$ is the Heaviside function. These correspond to step and rectangular pulses. We shall be interested in the response of the system (the surface of a semi-infinite homogeneous and isotropic body) to a δ-pulse thermal action:

$$q(\tau) = Q\delta(\tau), \tag{3}$$

where $\delta(\cdot)$ is the Dirac delta-function; Q is the energy density of the pulse. In order words, we will be interested in Green's time function or in the function of the temperature effect of instantaneous heat sources [3] for a boundary-value problem formulated in [1] for a semi-infinite body heated through a plane thin ring on the surface. This function is of practical interest for many problems of laser technology, since it gives a good approximation of the dynamics of a temperature field excited on the surface of a material by short (smaller than 1 msec) laser radiation pulses in a ring zone of illumination. The solution of the problem for a source of the form $q(\tau) = U(\tau) - U(\tau-\tau_0)$ is suitable for investigation of the thermophysical properties of materials under laboratory conditions, when one manages to realize rectangular heat pulses in practice [4]. A source of the form $q(\tau) = Q\delta(\tau)$ corresponds largely to the conditions of pulse heating of a material by radiation focused in a ring zone of illumination (see, for example, [5]).

Function of Temperature Effect of Instantaneous Heat Sources. At $q(\tau) = Q\delta(\tau)$ Eqs. (1) take the form:

$$\tilde{\Theta}_1 = \frac{R_1}{r_0} \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_{n,m} \left(\frac{r_0}{r_0}\right)^{2n} \left[\left(\frac{r}{R_1}\right)^{m+1/2} \exp\left(-\frac{x_1}{2}\right) S_{nm}(x_1) - \right.$$

$$\left. - \left(\frac{r_0}{R_2}\right)^{m+1/2} \exp\left(-\frac{x_2}{2}\right) S_{nm}(x_2)\right] ; \tag{4a}$$

$$\tilde{\Theta}_2 = \frac{R_2}{r_0} \left\{ \frac{1}{\sqrt{2}} - \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_{n,m} \left(\frac{r}{r_0}\right)^{2n} \left(\frac{r_0}{R_2}\right)^{m+1/2} \exp\left(-\frac{x_2}{2}\right) S_{nm}(x_2) - \right.$$

$$\left. - \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{A_{n,m}}{2(n+1)} \left(\frac{R_1}{r_0}\right)^{2(n+1)} \left(\frac{r_0}{r}\right)^{m+3/2} \exp\left(-\frac{x}{2}\right) S^1_{nm}(x) \right\} ; \tag{4b}$$

$$\tilde{\Theta}_3 = \frac{R_2}{r_0} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{A_{n,m}}{2(n+1)} \left(\frac{R_2}{r_0}\right)^{2(n+1)} \left[1 - \left(\frac{R_1}{R_2}\right)^{2(n+1)}\right] \left(\frac{r_0}{r}\right)^{m+3/2} \exp\left(-\frac{x}{2}\right) S^1_{nm}(x) . \tag{4c}$$

This is actually the function of the temperature effect of instantaneous heat sources for the problem considered. Here, the following notation is adopted:

$$\tilde{\Theta}_i = \sqrt{\pi/2}\, \frac{CR_i}{Q} \Theta_i(r, 0, \tau) , \quad S_{nm}(x) = W_{n-m/2+1/4, m/2-1/4}(x) ,$$

$$S^1_{nm}(x) = W_{n-m/2+3/4, m/2+1/4}(x) , \quad x = r^2/(4r_0^2) , \quad x_i = R_i^2/(4r_0^2) ; \tag{5}$$

$C = c\gamma$ is the volumetric heat capacity, and the coefficients $A_{n,m}$ are defined by formula (2). The functions $S^1_{nm}(x)$ and $S_{nm}(x)$ differ from each other by a shift of the indices (ν, μ) in the appropriate Whittaker functions $W_{\nu,\mu}$ by (1/2, 1/2). The time dependence of temperature in these relations is expressed in terms of the diffusion depth, i.e., the parameter

$$r_0 = \sqrt{a\tau} , \tag{6}$$

which determines the distance over which the isothermal surface is displaced in the body for the time τ due to diffusion.

At $R_1 = R_2$ all three expressions (4) vanish. For the functions $\Theta_1(r, 0, \tau)$ and $\Theta_3(r, 0, \tau)$ this is seen directly from Eqs. (4a) and (4c). For $\Theta_2(r, 0, \tau)$ this becomes evident from a consideration of its Laplace transform

$$\overline{\Theta}_2(r, z, s) = \frac{2R_2\bar{q}(s)}{\pi\lambda} \int_0^{\tau} \frac{\cos(pz)}{\sqrt{p^2 + s/p}} dp \left\{ \frac{1}{R_2\sqrt{p^2 + s/p}} - \right.$$

$$- \left[I_0(r\sqrt{p^2 + s/p})\, K_1(R_2\sqrt{p^2 + s/p}) + \left(\frac{R_1}{R_2}\right) \times \right.$$

$$\left.\left. \times I_1(R_1\sqrt{p^2 + s/p})\, K_0(r\sqrt{p^2 + s/p}) \right] \right\} , \quad R_1 \le r \le R_2 ,$$

with allowance for the formula for the Wronskian of the Bessel functions [6]:

$$W\left\{K_\nu(z),\ I_\nu(z)\right\} = I_\nu(z)\,K_{\nu+1}(z) + I_{\nu+1}(z)\,K_\nu(z) = \frac{1}{z}.$$

The distribution of integral temperatures can be found with the help of the equation

$$\langle T_i(r_1 \le r \le r_2,\ 0,\ \tau) - T_0\rangle = \frac{2}{r_2^2 - r_0^2}\int_{r_1}^{r_2} \Theta_i(r,\ 0,\ \tau)\, r dr .$$

With the use of the tabulated integration formula from [7], Eqs. (4a)-(4c) yield

$$\langle\widetilde{\Theta}_1(r_1 \le r \le r_2,\ 0,\ \tau)\rangle = \frac{R_1}{r_0}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\zeta_n}{n+1}A_{n,m}\left[\left(\frac{r_2}{R_1}\right)^{2n}\left(\frac{r_0}{R_1}\right)^{m+1/2}\times\right.$$

$$\times\exp\left(-\frac{R_1^2}{8r_0^2}\right)S_{nm}\left(\frac{R_1^2}{4r_0^2}\right) - \left(\frac{r_2}{R_2}\right)^{2n}\left(\frac{r_0}{R_2}\right)^{m+1/2}\times$$

$$\left.\times\exp\left(-\frac{R_2^2}{8r_0^2}\right)S_{nm}\left(\frac{R_2^2}{4r_0^2}\right)\right],\quad \zeta_n = \frac{1-(r_1/r_2)^{2n+2}}{1-(r_1/r_2)^2};$$

$$\langle\widetilde{\Theta}_2(r_1 \le r \le r_2,\ 0,\ \tau)\rangle = \frac{R_2}{r_0}\left\{\sqrt{2} - \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\zeta_n}{n+1}A_{n,m}\left(\frac{r_2}{R_2}\right)^{2n}\times\right.$$

$$\times\left(\frac{r_0}{R_2}\right)^{m+1/2}\exp\left(-\frac{R_2^2}{8r_0^2}\right)S_{nm}\left(\frac{R_2^2}{4r_0^2}\right) - \frac{R_1^2}{(r_2^2 - r_1^2)}\times$$

$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{A_{n,m}}{n+1}\left(\frac{R_1}{r_0}\right)^{2n} r_0^{m+1/2}\left[r_1^{-(m+1/2)}\exp\left(-\frac{r_1^2}{8r_0^2}\right)\times\right.$$

$$\left.\left.\times S_{nm}^1\left(\frac{r_1^2}{4r_0^2}\right) - r_2^{-(m+1/2)}\exp\left(-\frac{r_2^2}{8r_0^2}\right)S_{nm}^1\left(\frac{r_2^2}{4r_0^2}\right)\right]\right\}; \tag{7}$$

$$\langle\widetilde{\Theta}_3(r_1 \le r \le r_2,\ 0,\ \tau)\rangle = \frac{A}{r_0}\,\frac{R_2^2}{(r_2^2 - r_1^2)}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{A_{n,m}}{n+1}\left(\frac{R_2^2}{r_0}\right)^{2n} r_0^{m+1/2}\times$$

$$\times\left[r^{-(m+1/2)}\exp\left(-\frac{r_1^2}{8r_0^2}\right)S_{nm}^1\left(\frac{r_1^2}{4r_0^2}\right) - r^{-(m+1/2)}\exp\left(-\frac{r_2^2}{8r_0^2}\right)S_{nm}^1\left(\frac{r_2^2}{4r_0^2}\right)\right].$$

Here, the temperature normalization is the same as in (4). The upper and lower limits of integration r_1 and r_2 for each region are selected separately, i.e., $0 \le r_1 < r_2 \le R_1$ for the first, $R_1 \le r_1 < r_2 < R_2$ for the second, and $R_2 \le r_1 < r_2 < \infty$ for the third region.

To calculate S_{nm} and S^1_{nm} at small and large values of x, we can use formulas of asymptotic expansion and the limiting formulas for the Tricomi function $U(a, b, z)$ [6], with which the Whittaker function is associated by the relationship

$$W_{k,\mu}(z) = \exp\left(-\frac{z}{2}\right) z^{1/2+\mu} U(1/2 + \mu - k, 1 + 2\mu, z),$$

$$k = b/2 - a, \quad \mu = b/2 - 1/2.$$

We will need, in particular, the relations

$$U(a, b, z) = z^{-a} \left\{ \sum_{n=0}^{N-1} \frac{(a)_n (1 + a - b)_n}{n!} (-z)^n + 0(|z|^{-N}) \right\} \sim$$

$$\sim z^{-a}{}_2F_0(a, 1 + a - b; -z^{-1})$$

and

$$W_{k,\mu}(x) \sim \exp\left(-\frac{x}{2}\right) x^k{}_2F_0(a, 1 + a - b; -x^{-1}), \tag{8}$$

which are valid at large values of $x \sim R_1^2/r_0^2$. We will also need the differentiation formulas [7, 8]

$$\frac{d}{dx}\left[x^{\pm\mu-1/2} \exp\left(-\frac{x}{2}\right) W_{k,\mu}(x)\right] = -x^{\pm\mu-1} \exp\left(-\frac{x}{2}\right) W_{k+1/2,\mu\pm1/2}(x), \tag{9}$$

$$\frac{d}{dx}\left[x^k \exp\left(-\frac{x}{2}\right) W_{k,\mu}(x)\right] = -x^{k-1}\exp\left(-\frac{x}{2}\right) W_{k+1,\mu}(x). \tag{10}$$

A Model of the Temperature Field on the Surface of a Body Heated in a Narrow Ring of Illumination by Laser Radiation Pulses. Approximating Relations. Assuming the laser radiation pulses to be short as compared with time τ from the start of the pulse to the moment of observation, we shall construct a model based on a solution of the problem for the case of instantaneous heat sources, i.e., on formulas (4a)-(4c).

To simplify manipulations, we shall avail ourselves of an approximation based on the smallness of the quantities

$$\delta = r - R_1 << R_1, \quad R_1 \le r \le R_2, \quad \Delta = R_2 - R_1 << R_1. \tag{11}$$

In the majority of technical applications associated, in particular, with measurement technique and laser technology these conditions are fulfilled [5]. Usually $\Delta/R_1 < 0.01$. It is also necessary that the following inequality be satisfied:

$$\Delta < r_0^2/R_1, \tag{12}$$

i.e., the moment of observation τ should not be too small.

Using the formula for expansion into a Taylor series and differentiation formula (9), we reduce Eqs. (4) with allowance for inequalities (11) and (12) to the form

$$\tilde{\Theta}_1^* = A \exp\left(-\frac{x_1}{2}\right) \left(\frac{R_1}{r_0}\right)^{3/2} \sum_{n=0}^{\infty} \left(\frac{r}{r_0}\right) \sum_{m=0}^{n} A_{n,m} \left(\frac{r_0}{R_1}\right)^m S^2_{nm}(x_1),$$

$$\tilde{\Theta}_2^* = A \exp\left(-\frac{x}{2}\right)\left(\frac{r}{r_0}\right)^{3/2}\sum_{n=0}^{\infty}\left(\frac{r}{r_0}\right)^{2n}\sum_{m=0}^{n} A_{n,m}\left(\frac{r_0}{r}\right)^{m} S_{nm}^{1}(x), \tag{13}$$

$$\tilde{\Theta}_3^* = A \exp\left(-\frac{x_2}{2}\right)\left(\frac{R_2}{r_0}\right)^{3/2}\sum_{n=0}^{\infty}\left(\frac{R_2}{r_0}\right)^{2n}\left(\frac{R_2}{r}\right)^{3/2}\sum_{m=0}^{n} A_{n,m}\left(\frac{r_0}{r}\right)^{m} S_{nm}^{1}(x).$$

Here

$$\tilde{\Theta}_i^* = \Theta_i/(E/CR^3),\quad R = (R_1 + R_2)/2,\quad A = \frac{1}{2\pi\sqrt{\pi}},$$

$$S_{nm}^{2}(x) = W_{n-m/2+3/4,m/2+3/4}(x). \tag{14}$$

The quantity $E = 2\pi R\Delta Q$ designates the total energy in the pulse of thermal action.

In a number of cases when the material is exposed to short laser radiation pulses through a ring of illumination, we are interested in the temperature field at the time moments for which the depth of diffusion r_0 is much smaller than the inner radius of the ring R_1. The condition

$$x_1 = R_1^2/4r_0^2 >> 1 \tag{15}$$

allows us to make a further simplification of formulas (13) with the aid of formula (8) for the asymptotic representation of the Whittaker functions. When $x_1 >> 1$, we have

$$S^1(x) \sim S^2(x) \sim x^{h-m/2+3/4}\exp\left(-\frac{x}{2}\right) + (-1)^n n! \tag{16}$$

In addition to the principal term of the asymptotic expansion, relation (16) also takes into account the portion of the remainder term, that is independent of x in an approximation of large values of n: $(-1)^n n!$. On substitution of (16) into (13), the series are curtailed, i.e., they become the series of expansion of the Bessel functions

$$I_0\left(\frac{rR_i}{2r_0^2}\right) = \sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{rR_i}{4r_0^2}\right)^{2n},$$

$$I_1\left(\frac{rR_i}{2r_0^2}\right) = \sum_{n=0}^{\infty}\frac{1}{(n+1)(n!)^2}\left(\frac{rR_i}{4r_0^2}\right)^{2n+1}$$

and of the exponent

$$\exp\left(-\frac{r^2}{4r_0^2}\right) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{r^2}{4r_0^2}\right).$$

In this case we obtain

$$\Theta_1^* = \frac{1}{4\pi\sqrt{\pi}}\left(\frac{R_1}{r_0}\right)^3 \exp\left[-(x_1 + x)\right] I_0\left(\frac{rR_1}{2r_0^2}\right),\quad 0 \le r \le R_1,$$

$$\Theta_2^* = \frac{1}{4\pi\sqrt{\pi}} \left(\frac{r}{r_0}\right)^3 \exp(-2x)\, I_0\left(\frac{r^2}{2r_0^2}\right), \quad R_1 \le r \le R_2, \tag{17}$$

$$\Theta_3^* = \frac{1}{4\pi\sqrt{\pi}} \left(\frac{R_2}{r_0}\right)^3 \exp[-(x + x_2)]\, I_1\left(\frac{rR_2}{2r_0^2}\right), \quad R_2 \le r < \infty,$$

or in an asymptotic approximation of the Bessel functions $I_0(x) \sim I_1(x) \sim [1/(\sqrt{2\pi x})]\,\overline{\exp}(x)$ Eq. (17) yields

$$\Theta_1^* = \frac{1}{4\pi^2} \left(\frac{R_1^{3/2}}{\sqrt{r}\, r_0}\right) \exp[-(R_1 - r)^2/(4r_0^2)], \quad 0 \le r \le R_1; \tag{18a}$$

$$\Theta_2^* = \frac{1}{4\pi^2} \left(\frac{r}{r_0}\right)^2, \quad R_1 \le r \le R_2; \tag{18b}$$

$$\Theta_3^* = \frac{1}{4\pi^2} \left(\frac{R_2^{3/2}}{\sqrt{r}\, r_0}\right) \exp[-(r - R_2)^2/(4r_0^2)], \quad R_2 \le r < \infty. \tag{18c}$$

Here $\Theta_i^* = \Theta_i/(E/CR^2r_0)$, $E = 2\pi R\Delta Q$, $\Delta = R_2 - R_1 << R_1$.

These equations represent the simplest but still accurate (at small values of Δ/R_1 and r_0/R_1) model of a temperature field excited on a material surface by short laser radiation pulses in a bundle of annular shape.

We cannot regard the derivation of Eqs. (18) as entirely rigorous. There is uncertainty in the estimate of the remainder term in the asymptotic formula (16). The associated error is small but can manifest itself in the summation of series (13). Therefore, we shall give another derivation of Eqs. (18) that is free of the indicated uncertainty.

We shall proceed from the solution in the space of transforms. For the first region $0 \le r \le R_1$ it is written in the form [2]:

$$\overline{\Theta}(r, 0, s) = \frac{2\overline{q}(s)}{\pi\lambda} [J_1(R_1) - J_1(R_2)],$$

where

$$J_1(R_i) = R_i \int_0^\infty \frac{K_i(R_i x)}{x} I_0(rx)\, dp, \quad x \equiv \sqrt{p^2 + s/a}.$$

For a narrow ring the solution can be simplified by expanding $J_1(R_2)$ into a Taylor series in $\Delta = R_2 - R_1 << R_1$. In this case we obtain

$$\overline{\Theta}_1(r, 0, s) = \frac{2\overline{q}(s)\,\Delta R_1}{\pi\lambda} \int_0^\infty K_0(R_1 x)\, I_0(rx)\, dp.$$

In an asymptotic approximation of large values of x

$$K_0(R_1 x)\, I_0(rx) \sim \frac{1}{2\sqrt{R_1 r}} \frac{1}{x} \exp[-(R_1 - r)x].$$

Since

$$\int_0^\infty \frac{\exp[-(R_1 - r)\sqrt{p^2 + a/s}]}{\sqrt{p^2 + a/s}} dp = K_0 \left[\frac{R_1 - r}{\sqrt{a}} \sqrt{s}\right]$$

(see [7] p. 346), then with allowance for the formula for the inverse Laplace transform

$$L^{-1} [K_0 (k \sqrt{s})] = \frac{1}{2\tau} \exp[-k^2/(4\tau)]$$

for the case of instantaneous sources $q(\tau) = Q\delta(\tau)$ we have

$$\Theta_1 (r, 0, \tau) = \frac{Q\Delta}{2\pi C r_0^2} \sqrt{\left(\frac{R_1}{r}\right)} \exp\left[-\frac{(R_1 - r)^2}{4 r_0^2}\right], \quad 0 \le r \le R_1 .$$

This relation coincides with Eq. (18a). Similarly we derive formulas (18b) and (18c).

Comparison with Known Results for a Narrow Ring and a Disk. Let us compare the results obtained with known solutions of the problem of heating of a semi-infinite body for two particular cases of exposure of the surface to instantaneous heat sources of axial symmetry. This is the case of an infinite thin ring and a disk [9].

1. An instantaneous heat source with pulse energy E acts on the surface of a body along a circle R at time $t = 0$. The temperature at the point with the coordinates r, θ, $\bar{z}$ at this time t is determined by the expression [9, 10]:

$$\Theta (r, z, t) = \frac{(E/C)}{4 (\pi a t)^{3/2}} \frac{1}{2\pi} \int_0^{2\pi} \exp\left[-\frac{r^2 + z^2 + R^2 - 2rR\cos(\Theta - \Theta')}{4at}\right] d\Theta' =$$

$$= \frac{(E/C)}{4 (\pi a t)^{3/2}} \exp\left(-\frac{r^2 + R^2 + z^2}{4at}\right) I_0 \left(\frac{rR}{2at}\right), \tag{19}$$

where $I(\cdot)$ is a modified Bessel function; C is the volumetric heat capacity of the body. This expression is obtained by integration of the fundamental solution (Green function) over the circumference for an instantaneous point source with coordinates x', y', z' on the surface of a semi-infinite body:

$$v (x, y, z, t) = \frac{(Q/C)}{4 (\pi a t)^{3/2}} \exp\left\{-\frac{(x - x')^2 + (y - y')^2 + (z - z')^2}{4at}\right\}.$$

For integration we assume that the energy of the point source is $QRd\theta$ and the total energy in the ring $E = 2\pi RQ$.

With account for the asymptotic formula

$$I_0 (x) \sim \frac{\exp(x)}{\sqrt{2\pi x}}$$

Eq. (19) for small values of r_0^2/rR yields

$$\Theta (r, z, t) = \frac{(E/C)}{4\pi^2 r_0^2 \sqrt{rR}} \exp\left(-\frac{z^2}{4r_0^2}\right) \exp\left[-\frac{(r - R)}{4r_0^2}\right], \tag{20}$$

where r_0 is the depth of heat diffusion determined from formula (6). At $R = R_2$ and $z = 0$, Eq. (20) gives Eq. (18b).

2. An instantaneous source acts over a disk of radius R. For this case the solution is obtained from Eq. (19) by integration with respect to r' after replacing R by r' and E by $2\pi Q r' dr'$ in this equation. The solution has the form

$$\Theta(r, z, t) = \frac{(E/C)}{2R^2 (\pi a t)^{3/2}} \int_0^R \exp\left[-\frac{r^2 + r'^2 + z^2}{4at}\right] I_0\left(\frac{rr'}{2at}\right) r' dr' =$$

$$= \frac{(E/C)}{2R(\pi a t)^{3/2}} \exp\left(-\frac{z^2}{4at}\right) \int_0^\infty \exp(-at\lambda^2) J_0(\lambda r) J_1(\lambda R)\, d\lambda .$$

In general, the integral is not expressed in tabular functions; however, at $r = R$ it can be calculated. The solution takes the form

$$\Theta(R, z, t) = \frac{(E/C)}{\pi^{3/2} R^2 r_0} \left[1 - \exp\left(-\frac{R^2}{2r_0^2}\right)\right] \exp\left(-\frac{z^2}{4r_0^2}\right) . \tag{21}$$

Formula (21) agrees with formula (4b). By performing the limiting transition $\tau \to 0$ and simple transformations, from Eq. (4b) at $R_1 = 0$ and $r = R_2 = R$ for $z = 0$ we obtain

$$\tilde{\Theta}_2(\tau \to 0) = \frac{R}{r_0} \frac{1}{\sqrt{2}} \Rightarrow \Theta_2 = \frac{(E/C)}{\pi^{3/2} R^2 r_0} .$$

This equation coincides with Eq. (21) for $z = 0$ and $r \to 0$.

NOTATION

$\Theta_i(r, 0, \tau) = T_i(r, z, \tau) - T_0$ at $z = 0$, excess temperatures of semi-infinite body in corresponding regions where the quantity r varies (see text); T_0, initial temperature; R_2, R_1, r, outer and inner radii of the ring heat source and the instantaneous radius; z, τ, instantaneous (cylindrical) coordinate normal to the surface of semi-infinite body and current time; p, s, parameters of Fourier infinite integral cosine transform and of Laplace transform; $q(\tau)$, arbitrary (in time) density of a heat flux in a given local region of heating for a source; a, λ, b, thermal diffusivity, thermal conductivity, and thermal activity; C, volumetric heat capacity of semi-infinite body equal to $c\gamma$, where c and γ are the specific heat capacity and density of the material; $A_{n,m}$, constant thermal amplitudes (see text); $U(\tau)$, unit step Heaviside function; $W_{k,\mu}(x)$, Whittaker function; $F_0(a, b, x)$, generalized hypergeometric function; $I_\nu(\cdot)$ and $K_\nu(\cdot)$, modified Bessel functions of the first and third kind of the nth order; $\delta(\tau)$, Dirac delta-function.

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